INVESTIGATION OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR A COMPOSITE TYPE EQUATION WITH NON-LOCAL BOUNDARY CONDITIONS

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Abstract. Since the order of elliptic type model equation (Laplace equation) is two [1], [2], then it is natural the order of composite type model equation must be [3] [4] [5] three. At each point of the domain under consideration these equations have both real and complex characteristics.

Notice that a boundary value problem for a composite type equation of second order first appeared in the paper [6].

The method for investigating the Fredholm property of boundary value problems is distinctive and belongs to one of the authors of the present paper.
Key words: Composite type equations, non local boundary conditions for partial differential equations, fundamental solution, necessary condition, regularization, Fredholm property.

Introduction.

The paper is devoted to the investigation of boundary value problems for a composite type equation of second order.

This was possible as earlier we investigated an elliptic type boundary value problem of first order (Cauchy-Riemann equation) for which the boundary is a carrier of boundary conditions [7]. Notice that in this case it is impossible to determine local boundary conditions (since undeterminacy is obtained). Therefore non-local boundary conditions were considered.

Necessary conditions that contain singular integrals are obtained proceeding from fundamental solution of Cauchy-Riemann equation [8]. Considering that we are on a spectrum, regularization of these singularities is also conducted in distinctive way [6]. Joining regularized necessary conditions with the given boundary conditions we get a sufficient condition for Fredholm property of the stated boundary value problems.

Notice that in [8] the cited investigation of the process in a nuclear reactor leads to a boundary value problem for first order integro-differential equation in three-dimensional space where not all the space is a carrier of the given local boundary condition.

Problem statement

Let's consider the following boundary value problem:

$$\ell\, u \equiv \frac{\partial^2 u(x)}{\partial x_2^2} + i\frac{\partial^2 u(x)}{\partial x_1\,\partial x_2} = 0, \quad x \in D, \tag{1}$$

$$\ell_k u \equiv \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_k(x_1)} + \alpha_k u(x_1, \gamma_k(x_1)) = \varphi_k(x_1), \quad k = 1,\, 2; \; x_1 \in [a_1, b_1] \tag{2}$$

where $i = \sqrt{-1}$, $D \subset R^2$- is a bounded domain convex in the direction x_2, the boundary $\Gamma = \bar{D} \backslash D$ - is a Liapunov line, $\gamma_k(x_1)$, $k = 1,\, 2$ are the equations of open lines Γ_k ($\Gamma_1 \bigcup \Gamma_2 = \Gamma$), obtained from the boundary Γ of the domain D by means of orthogonal projection of this domain on the axis x_1 parallel to the axis x_2 and $[a_1,\, b_1] = np_{x_1}\Gamma_1 = np_{x_1}\Gamma_2$. In the given boundary conditions (17) α_k ($k = 1,\, 2$) are constants, $\varphi_k(x_1)$, $k = 1,\, 2$; $x_1 \in [a_1,\, b_1]$ are sufficiently smooth functions. Boundary conditions (17) are assumed to be linear independent .

Fundamental solution.

Applying the Fourier transform to equation (16) we get a fundamental solution in the form

$$U(x-\xi) = \frac{-1}{4\pi^2} \int_{R^2} \frac{e^{i(\alpha,\, x-\xi)}}{\alpha_2(\alpha_2 + i\, \alpha_1)} d\alpha, \tag{3}$$

where

$$(\alpha_1 x - \xi) = \sum_{j=1}^{2} \alpha_j (x_j - \xi_j).$$

Further, since

$$\frac{1}{2\pi i} \int_R \frac{e^{i\alpha_2(x_2-\xi_2)}}{\alpha_2} d\alpha_2 = e(x_2 - \xi_2),$$

where $e(x_2 - \xi_2)$ is a unique symmetric Heaviside function, from (3) we get:

$$\frac{\partial U(x-\xi)}{\partial x_2} + i \frac{\partial U(x-\xi)}{\partial x_2} = e(x_2 - \xi_2)\, \delta(x_1 - \xi_1). \tag{4}$$

Finally considering that (3) is a fundamental solution of the composite type equation (16), we get that $\frac{\partial U(x-\xi)}{\partial x_2}$ is a fundamental solution of the Cauchy-Riemann equation. Making negligible changes in the fundamental solution of the Cauchy-Riemann equation [8] we get a fundamental solution in the direction x_2

$$\frac{\partial U(x-\xi)}{\partial x_2} = \frac{1}{2\pi} \frac{\theta(x_2 - \xi_2) + \theta(\xi_2 - x_2)}{x_2 - \xi_2 + i(x_1 - \xi_1)}. \tag{5}$$

here $\theta(x_2 - \xi_2) + \theta(x_2 - \xi_2) = 1$, if none differentiation operation is produced on it, since each addend has a break and contribution of this break appears in differentiation. Thus, for fundamental solution (3) of composite type equation (16) we get:

$$U(x-\xi) = \frac{1}{2\pi} \int_0^{x_2} \frac{\theta(t - \xi_2) + \theta(\xi_2 - t)}{t - \xi_2 + i(x_1 - \xi_1)} dt, \tag{6}$$

i.e. it holds the following statement:

Theorem 1. For a composite type equation of second order (16) a fundamental solution in the direction x_2 is of the form (6).

This means that if we differentiate $U(x-\xi)$ twice with respect to x_2 and twice with respect to the mixed derivatives x_1 and x_2, the Dirac delta function (two-dimensional) appears only in the derivative of second order with respect to x_2.

Necessary conditions.

Multiplying equation (16) by fundamental solution (6) and integrating if in domain D, applying Ostrogradskii-Gauss formula [8], we get formula similar to Green's second formula that after application of fundamentality properties of function (6) get the form:

$$\int_{a_1}^{b_1} \left[u(x)\frac{\partial U(x-\xi)}{\partial x_2} - \frac{\partial u(x)}{\partial x_2} U(x-\xi) \right] \Bigg|_{x_2=\gamma_1(x_1)}^{\gamma_2(x_1)} dx_1 + i \int_{a_1}^{b_1} u(x)\frac{\partial U(x-\xi)}{\partial x_1} \Bigg|_{x_2=\gamma_1(x_1)}^{\gamma_2(x_1)} dx_1 +$$

$$+ i \int_{a_1}^{b_1} \frac{\partial u(x)}{\partial x_2} U(x-\xi) \Bigg|_{x_2=\gamma_2(x_1)} \gamma_2'(x_1) dx_1 - i \int_{a_1}^{b_1} \frac{\partial U(x)}{\partial x_2} U(x-\xi) \Bigg|_{x_2=\gamma_1(x_1)} \gamma_2'(x_1) dx_1 =$$

$$= \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2} u(\xi), & \xi \in \Im \end{cases} \tag{7}$$

The second expression in formula (7) is one of the necessary conditions. This condition has the form:

$$u(\xi_1, \gamma_1(\xi_1)) = u(\xi_1, \gamma_2(\xi_1)) - 2\int_{a_1}^{b_1} \frac{\partial u(x)}{\partial x_2}\Bigg|_{x_2=\gamma_2(x_1)} U(x_1-\xi_1, \gamma_2(x_1)-\gamma_1(\xi_1)) \left[1 - i\gamma_2'(x_1)\right] dx_1 +$$

$$+ 2\int_{a_1}^{b_1} \frac{\partial u(x)}{\partial x_2}\Bigg|_{x_2=\gamma_1(x_1)} U(x_1-\xi_1, \gamma_1(x_1)-\gamma_1(\xi_1)) \left[1 - i\gamma_1'(x_1)\right] dx_1 \tag{8}$$

In exactly the same way to [9] and [10], we get the following necessary conditions:

$$\frac{\partial u(\xi)}{\partial \xi_1}\Bigg|_{\xi_2=\gamma_1(\xi_1)} = \frac{\partial u(\xi)}{\partial \xi_1}\Bigg|_{\xi_2=\gamma_2(\xi_1)} - i\,\frac{\partial u(\xi)}{\partial \xi_2}\Bigg|_{\xi_2=\gamma_2(\xi_1)} -$$

$$-2i \int_{a_1}^{b_1} \frac{\partial u(x)}{\partial x_2}\Bigg|_{x_2=\gamma_1(x_1)} \frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_1(\xi_1)}} \left[1 - i\gamma_1'(x_1)\right] dx_1 +$$

$$+ 2i \int_{a_1}^{b_1} \frac{\partial u(x)}{\partial x_2}\Bigg|_{x_2=\gamma_2(x_1)} \frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_1(\xi_1)}} \left[1 - i\gamma_2'(x_1)\right] dx_1, \tag{9}$$

$$\frac{\partial u(\xi)}{\partial \xi_2}\Bigg|_{\xi_2=\gamma_1(\xi_1)} = 2\int_{a_1}^{b_1} \frac{\partial u(x)}{\partial x_2}\Bigg|_{x_2=\gamma_2(x_1)} \frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_1(\xi_1)}} \left[1 - i\gamma_2'(x_1)\right] dx_1 -$$

$$- 2\int_{a_1}^{b_1} \frac{\partial u(x)}{\partial x_2}\Bigg|_{x_2=\gamma_1(x_1)} \frac{\partial U(x-\xi)}{\partial x_2}\Bigg|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_1(\xi_1)}} \left[1 - i\gamma_1'(x_1)\right] dx_1, \tag{10}$$

$$\frac{\partial u(\xi)}{\partial \xi_1}\Bigg|_{\xi_2=\gamma_2(\xi_1)} = \frac{\partial u(\xi)}{\partial \xi_1}\Bigg|_{\xi_2=\gamma_1(\xi_1)} - i\,\frac{\partial u(\xi)}{\partial \xi_2}\Bigg|_{\xi_2=\gamma_1(\xi_1)} -$$

$$-2i\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}\left.\frac{\partial U(x-\xi)}{\partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_2(\xi_1)}}[1-i\gamma_1'(x_1)]\,dx_1+$$

$$+2i\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}\left.\frac{\partial U(x-\xi)}{\partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_2(\xi_1)}}[1-i\gamma_2'(x_1)]\,dx_1, \qquad (11)$$

$$\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_2(\xi_1)}=2\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}\left.\frac{\partial U(x-\xi)}{\partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_2(\xi_1)}}[1-i\gamma_2'(x_1)]\,dx_1-$$

$$-2\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}\left.\frac{\partial U(x-\xi)}{\partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_2(\xi_1)}}[1-i\gamma_1'(x_1)]\,dx_1. \qquad (12)$$

Thus, we established the following statement:

Theorem 2. Let D be a plane domain convex in the direction x_2, the boundary Γ be Liapunov line, then each solution of equation (16) determined in the domain D satisfies the necessary conditions (8)–(12), containing singular integrals besides (8).

Regularization.

As it was said above, necessary conditions (9)–(12) contain singular addends. Considering (5), we have:

$$\left.\frac{\partial U(x-\xi)}{\partial x_2}\right|_{\substack{x_2=\gamma_k(x_1)\\ \xi_2=\gamma_k(\xi_1)}}=\frac{1}{2\pi}\cdot\frac{1}{\gamma_k(x_1)-\gamma_k(\xi_1)+i(x_1-\xi_1)}=$$

$$=\frac{1}{2\pi}\frac{1}{x_1-\xi_1}\cdot\frac{1}{\gamma_k'(\sigma_k(x_1,\,\xi_1))+i},\,k=1,\,2,$$

where $\sigma_k(x_1,\,\xi_1)$ is located between x_1 and ξ_1. Then from (9) – (12) we find:

$$\left.\frac{\partial u(\xi)}{\partial \xi_1}\right|_{\xi_2=\gamma_1(\xi_1)}-\left.\frac{\partial u(\xi)}{\partial \xi_1}\right|_{\xi_2=\gamma_2(\xi_1)}+i\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_2(\xi_1)}=$$

$$=-\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}\frac{dx_1}{x_1-\xi_1}+\ldots\ldots,$$

$$\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_1(\xi_1)}=\frac{i}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}\frac{dx_1}{x_1-\xi_1}+\ldots\ldots, \qquad (13)$$

$$\left.\frac{\partial u(\xi)}{\partial \xi_1}\right|_{\xi_2=\gamma_2(\xi_1)}-\left.\frac{\partial u(\xi)}{\partial \xi_1}\right|_{\xi_2=\gamma_1(\xi_1)}+i\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_1(\xi_1)}=$$

$$=\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}\frac{dx_1}{x_1-\xi_1}+\ldots\ldots,$$

$$\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_2(\xi_1)}=-\frac{i}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}\frac{dx_1}{x_1-\xi_1}+\ldots\ldots$$

where the sum of non-singular addends are denoted by dots.

Considering boundary conditions (17), from necessary conditions (8) for boundary values of the unknown function we get the following regular relations:

$$u(\xi_1, \gamma_1(\xi_1) = u(\xi_1,\, \gamma_2(\xi_1)) -$$
$$-2\int_{a_1}^{b_1} [\varphi_2(x_1) - \alpha_2 u(x_1,\, \gamma_2(x_1)]\, U(x_1 - \xi_1,\, \gamma_2(x_1) - \gamma_1(\xi_1))\, [1 - i\gamma_2'(x_1)]\, dx_1 +$$

$$+\, 2\int_{a_1}^{b_1} [\varphi_1(x_1) - \alpha_1 u(x_1,\, \gamma_1(x_1)]\, U(x_1 - \xi_1,\, \gamma_1(x_1) - \gamma_1(\xi_1))\, [1 - i\gamma_1'(x_1)]\, dx_1, \qquad (14)$$

In exactly the same way, from (13) we get:

$$\varphi_1(\xi_1) - \alpha_1 u(\xi_1,\, \gamma_1(\xi_1)) = \frac{i}{\pi}\int_{a_1}^{b_1} [\varphi_1(x_1) - \alpha_1 u(x_1, \gamma_1(x_1))]\, \frac{dx_1}{x_1 - \xi_1} + \ldots\ldots,$$

$$\varphi_2(\xi_1) - \alpha_2 u(\xi_1,\, \gamma_2(\xi_1)) = -\frac{i}{\pi}\int_{a_1}^{b_1} [\varphi_2(x_1) - \alpha_2 u(x_1, \gamma_2(x_1))]\, \frac{dx_1}{x_1 - \xi_1} + \ldots\ldots.$$

Finally, proceeding from (14) for boundary values of the unknown function we get the following regular relation [9],[10]

$$\frac{\varphi_1(\xi_1)}{\alpha_1} + \frac{\varphi_2(\xi_1)}{\alpha_2} - [u(\xi_1, \gamma_1(\xi_1)) + u(x_1, \gamma_2(\xi_1)] = \frac{i}{\pi}\int_{a_1}^{b_1}\left[\frac{\varphi_1(x_1)}{\alpha_1} - \frac{\varphi_2(x_1)}{\alpha_2}\right]\frac{dx_1}{x_1 - \xi_1} -$$
$$-\frac{i}{\pi}\int_{a_1}^{b_1}\Big\{-2\int_{a_1}^{b_1} [\varphi_2(\eta_1) - \alpha_2 u(\eta_1, \gamma_2(\eta_1))]\; U(\eta_1 - x_1,\, \gamma_2(\eta_1) - \gamma_1(x_1))\, [1 - i\gamma_2'(\eta_1)]\; d\eta_1 +$$

$$+2\int_{a_1}^{b_1} [\varphi_1(\eta_1) - \alpha_1 u(\eta_1, \gamma_1(\eta_1)]\; U(\eta_1 - x_1,\, \gamma_1(\eta_1) - \gamma_1(x_1))\, [1 - i\gamma_1'(\eta_1)]\; d\eta_1\Big\}\, \frac{dx_1}{x_1 - \xi_1} + \ldots \qquad (15)$$

that is regular if we interchange the integrals contained in the right hand side of (15) and consider the singular integrals of unknown functions calculated in [11]. Thus we proved the following statement:

Theorem 3. When fulfilling the conditions of Theorem 2 if $\varphi_k(x_1)$, $k = 1, 2$ are continuously differentiable functions vanishing at the end of the interval $(a_1,\, b_1)$, then (15) are regular relations.

Fredholm property.

Considering boundary conditions (17), the first necessary condition (8) not containing singular integrals leads to regular relation (14).

Further, proceeding from boundary conditions, after regularizing two necessary conditions given in (13), that contain singular integrals, we get a relation that has no singularity in the form (15).

It holds :

Theorem 4. When fulfilling conditions of Theorem 3 boundary value problem (16)–(17) is Fredholm.

Really, it is easy to get from (14) and (15) a system of Fredholm integral equations of second kind with respect to the unknown functions $u(x_1, \gamma_k(x_1))$, $k = 1, 2$, in which a kernel may have only weak singularity.

Unsolved problems.

1. The inverse problem in Tikhonov-Lavrent'ev sense.

Let's consider the problem

$$\frac{\partial^2 u(x)}{\partial x_2^2} + i\frac{\partial^2 u(x)}{\partial x_1 \partial x_2} = 0, \quad x \in D, \tag{16}$$

$$\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_k(x_1)} + \alpha_k u(x_1, \gamma_k(x_1)) = \varphi_k(x_1), \quad k = 1,\, 2;\; x_1 \in [a_1,\, b_1]\,, \tag{17}$$

with the following complementary restriction

$$\alpha_1(x_1)\left.\tfrac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} + \alpha_2(x_1)\left.\tfrac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} +$$
$$+\alpha_3 u(x_1, \gamma_1(x_1)) + \alpha_4(x_1)\, u(x_1,\, \gamma_2(x_1)) = \varphi_3(x_1), \quad x_1 \in [a_1,\, b_1]$$

where , , $\varphi_1(x_1)$, , $k = 1,\, 3,\, 4$ and are the known, $u(x)$, $x \in D$ $\varphi_2(x_1) = \alpha_2(x_1)$ - are the unknown functions.

2. Stephan's inverse problem.

The above mentioned boundary value problem (16),(17), is given provided $\alpha_k,\ \varphi_k(x_1)$, $k = 1,\, 2$, $\gamma_1(x_1)$, $\alpha_k(x_1)$, $k = \overline{1,\, 4}$ $\varphi_3(x_1)$ - are the known, $u(x)$, $x \in D$ $\gamma_2(x_1)$, $x_1 \in [a_1,\, b_1]$ are the unknown functions.

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